Paper Reference(s)

6676/01

Edexcel GCE

Further Pure Mathematics FP3

Advanced Level

Tuesday 26 June 2007 − Afternoon

Time: 1 hour 30 minutes

Materials required for examination	**Items included with question papers**
Mathematical Formulae (Green)	Nil

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulas stored in them.

Instructions to Candidates

Write the name of the examining body (Edexcel), your centre number, candidate number, the unit title (Further Pure Mathematics FP3), the paper reference (6676), your surname, initials and signature.

Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
There are 8 questions in this question paper. The total mark for this paper is 75.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner. Answers without working may gain no credit.

1.
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\,\mathrm{e}^{x^2}.$$

It is given that $y = 0.2$ at $x = 0$.

(*a*) Use the approximation $\dfrac{y_1 - y_0}{h} \approx \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0$, with $h = 0.1$, to obtain an estimate of the value of y at $x = 0.1$.

(2)

(*b*) Use your answer to part (*a*) and the approximation $\dfrac{y_2 - y_0}{2h} \approx \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_1$, with $h = 0.1$, to obtain an estimate of the value of y at $x = 0.2$.

Give your answer to 4 decimal places.

(3)

2.
$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

At $x = 0$, $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$.

(*a*) Find the value of $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ at $x = 0$.

(3)

(*b*) Express y as a series in ascending powers of x, up to and including the term in x^3.

(4)

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3. Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix},$$

(*a*) find the eigenvalue of **A** corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

(2)

(*b*) find the value of p and the value of q.

(4)

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by **A** is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

(*c*) Using the values of p and q from part (*b*), find the values of the constants l, m and n.

(4)

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4. (*a*) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(2)

(*b*) Express $32\cos^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where p, q, r and s are integers.

(5)

(*c*) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \cos^6\theta \; \mathrm{d}\theta$.

(4)

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5. Prove by induction that, for $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

(5)

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6. Given that $$f(n) = 3^{4n} + 2^{4n+2},$$

(*a*) show that, for $k \in \mathbb{Z}^+$, $f(k+1) - f(k)$ is divisible by 15,

(4)

(*b*) prove that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 5,

(3)

(*c*) show that it is not true that, for all positive integers n, $f(n)$ is divisible by 15.

(1)

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7. The points A, B and C have position vectors, relative to a fixed origin O,

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j},$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

$$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

respectively. The plane Π passes through A, B and C.

(*a*) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(4)

(*b*) Show that a cartesian equation of Π is $3x - y + 2z = 7$.

(2)

The line l has equation $(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. The line l and the plane Π intersect at the point T.

(*c*) Find the coordinates of T.

(5)

(*d*) Show that A, B and T lie on the same straight line.

(3)

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8. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + \mathrm{i}}{z}, \quad z \neq 0.$$

(*a*) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than (0, 0), to points on a line l in the w-plane. Find a cartesian equation of l.

(5)

(*b*) Show that the image, under T, of the line with equation $x + y + 1 = 0$ □□□in the z-plane is a circle C in the w-plane, where C has cartesian equation

$$u^2 + v^2 - u + v = 0.$$

(7)

(*c*) On the same Argand diagram, sketch l and C.

(3)

TOTAL FOR PAPER: 75 MARKS

END

June 2007
6676 Further Pure Mathematics FP3
Mark Scheme

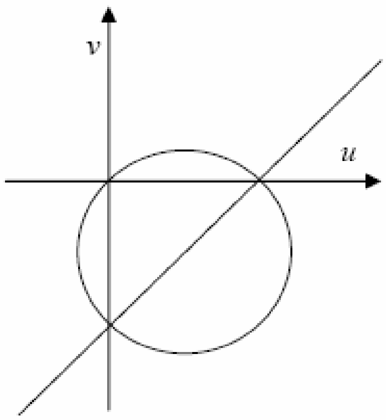
Question Number	Scheme	Marks
1.	(a) $\frac{y_1 - 0.2}{0.1} \approx \left(\frac{dy}{dx}\right)_0 = 0.2 \times e^0 \quad (= 0.2)$	M1
	$y_1 \approx 0.22$	A1 **(2)**
	(b) $\left(\frac{dy}{dx}\right)_1 \approx 0.22 \times e^{0.01} \approx 0.2222 \ \ldots$	B1
	$\frac{y_2 - 0.2}{0.2} \approx 0.2222 \ldots$	M1
	$y_2 \approx 0.2444$ cao	A1 **(3)**
		[5]
2.	(a) $\left(1-x^2\right)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2\frac{dy}{dx} = 0$	M1
	At $x = 0$, $\frac{d^3y}{dx^3} = -\frac{dy}{dx} = 1$	M1 A1 cso **(3)**
	(b) $\left(\frac{d^2y}{dx^2}\right)_0 = -4$ Allow anywhere	B1
	$y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ \ldots$	
	$= 2 - x - 2x^2, + \frac{1}{6}x^3 + \ldots$	M1 A1ft, A1 (dep)**(4)**
		[7]

Question Number	Scheme	Marks
3.	(a) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	
	Third row $\quad 1-3=-\lambda \quad \Rightarrow \quad \lambda=2$	M1 A1 **(2)**
	(b) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	M1 A1
	First row $\quad 4-p=0 \quad \Rightarrow \quad p=4$ Method for either	M1
	Second row $\quad q+4=2 \quad \Rightarrow \quad q=-2$ Both correct	A1 ft **(4)**
	(c) $\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix}.\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$	
	$3l+4m+4n=10$ $-l-2m-4n=-4$ $l+m+3n=3$ Obtaining 3 linear equations	M1
	$2l+2m=6$ $3l+2m=8$ Reducing to a pair of equations and solving for one variable	M1
	$l=2, m=1, n=0$ Solving for all three variables.	M1 A1 **(4)** **[10]**
	Alternative to (c)	
	$\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix}$	M1 M1
	$\frac{1}{6}\begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix}.\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	M1 A1 **(4)**

Question Number	Scheme	Marks
4.	(a) $z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$	
	$z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$ both	M1
	Adding $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ ∗ cso	A1 **(2)**
	(b) $\left(z + \dfrac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$	M1
	$= z^6 + z^{-6} + 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) + 20$	M1
	$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M1
	$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$	A1, A1
	$(p = 1, q = 6, r = 15, s = 10)$ A1 any two correct	**(5)**
	(c) $\int \cos^6\theta \,\mathrm{d}\theta = \left(\dfrac{1}{32}\right)\int (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$	
	$= \left(\dfrac{1}{32}\right)\left[\dfrac{\sin 6\theta}{6} + \dfrac{6\sin 4\theta}{4} + \dfrac{15\sin 2\theta}{2} + 10\theta\right]$	M1 A1ft
	$[\ \ldots\ ]_0^{\frac{\pi}{3}} = \dfrac{1}{32}\left[-\dfrac{3}{2}\times\dfrac{\sqrt{3}}{2} + \dfrac{15}{2}\times\dfrac{\sqrt{3}}{2} + \dfrac{10\pi}{3}\right] = \dfrac{5\pi}{48} + \dfrac{3\sqrt{3}}{32}$ or exact equivalent	M1 A1 **(4)**
		[11]

Question Number	Scheme	Marks
5.	$n=1$: $\quad 1^2 = \frac{1}{3}\times1\times1\times3$	B1
	(Hence result is true for $n=1$.)	
	$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2$	
	$= \frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$, by induction hypothesis	M1
	$= \frac{1}{3}(2k+1)(2k^2-k+6k+3)$	
	$= \frac{1}{3}(2k+1)(2k^2+5k+3)$	
	$= \frac{1}{3}(2k+1)(2k+3)(k+1)$	M1 A1
	$= \frac{1}{3}(k+1)[2(k+1)-1][2(k+1)+1]$	
	(Hence, if result is true for $n=k$, then it is true for $n=k+1$.)	
	By Mathematical Induction, above implies the result is true for all $n\in\mathbb{Z}^+$. ∗	
	cso	A1 (5)
		[5]
6.	(a) $f(k+1)-f(k) = 3^{4k+4}+2^{4k+6}-3^{4k}-2^{4k+2}$	
	$= 3^{4k}(3^4-1)+2^{4k+2}(2^4-1)$	M1
	$= 3^{4k}\times80+2^{4k+2}\times15$ can be implied	A1
	$= 3^{4k-1}\times240+2^{4k+2}\times15 = 15(16\times3^{4k-1}+2^{4k+2})$	M1
	Hence $15 \mid f(k+1)-f(k)$ ∗ cso	A1 (4)
	Note: $f(k+1)-f(k)$ is divisible by 240 and other appropriate multiples of 15 lead to the required result.	
	(b) $n=1$: $\quad f(1)=3^4+2^6=145=5\times29 \Rightarrow 5 \mid f(1)$	B1
	(Hence result is true for $n=1$.)	
	From (a) $f(k+1)-f(k)=15\lambda$, say. By induction hypothesis $f(k)=5\mu$, say.	
	$f(k+1)=f(k)+15\lambda=5(\mu+3\lambda) \Rightarrow 5 \mid f(k+1)$	M1
	(Hence, if result is true for $n=k$, then it is true for $n=k+1$.)	
	By Mathematical Induction, above implies the result is true for all $n\in\mathbb{Z}^+$. ∗	
	Accept equivalent arguments cso	A1 (3)
	(c) $f(1)=145=5\times29$ is not divisible by 15, so result is not true for all $\mathbb{Z}^+$.	B1 (1)
	Note: There is no integer for which $f(n)$ is divisible by 15 and any specific example should be accepted.	[8]

Question Number	Scheme	Marks
7.	(a) $\overrightarrow{AB} = -\mathbf{i}+3\mathbf{j}+3\mathbf{k}, \quad \overrightarrow{AC} = 4\mathbf{j}+2\mathbf{k}$ any two	B1
	$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} = -6\mathbf{i}+2\mathbf{j}-4\mathbf{k}$	M1 A1 A1
	Give A1 for any two components correct or the negative of the correct answer.	**(4)**
	(b) Cartesian equation has form $3x - y + 2z = p$	
	$(2,-1,0) \Rightarrow 6+1 = p$ or use of another point	M1
	$3x - y + 2z = 7$ ✱ or any multiple	A1 **(2)**
	(c) Parametric form of line is $\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ or equivalent form	M1 A1
	Substituting into equation of plane	
	$3(5+2\lambda)-(5-\lambda)+2(3-2\lambda)=7$	M1
	Leading to $\lambda = -3$	A1
	$T:(-1,8,9)$	A1 **(5)**
	(d) $\overrightarrow{AT} = -3\mathbf{i}+9\mathbf{j}+9\mathbf{k}, \quad \overrightarrow{BT} = -2\mathbf{i}+6\mathbf{j}+6\mathbf{k}$ both	M1
	These are parallel and hence A, B and T are collinear ✱ (by the axiom of parallels)	M1 A1 **(3)** **[14]**
	Alternative to (d)	
	The equation of AB: $\mathbf{r} = 2\mathbf{i}-\mathbf{j}+\mu(-\mathbf{i}+3\mathbf{j}+3\mathbf{k})$ or equivalent	
	$\mathbf{i}$: $-1 = 2-\mu \Rightarrow \mu = 3$	M1
	$\mu = 3 \Rightarrow \overrightarrow{OT} = -\mathbf{i}+8\mathbf{j}+9\mathbf{k}$	M1
	Hence A, B and T are collinear ✱ cso	A1 **(3)**
	Note: Column vectors or bold-faced vectors may be used at any stage.	

Question Number	Scheme	Marks
8.	(a) Let $z=\lambda+\lambda\mathrm{i}$; $\quad w=\dfrac{\lambda+(\lambda+1)\mathrm{i}}{\lambda(1+\mathrm{i})}$	M1
	$=\dfrac{\lambda+(\lambda+1)\mathrm{i}}{\lambda(1+\mathrm{i})}\times\dfrac{1-\mathrm{i}}{1-\mathrm{i}}$	M1
	$u+\mathrm{i}v=\dfrac{(2\lambda+1)+\mathrm{i}}{2\lambda}$	A1
	$u=1+\dfrac{1}{2\lambda},\quad v=\dfrac{1}{2\lambda}$	M1
	Eliminating λ gives a line with equation $v=u-1$ or equivalent	A1 (5)
	(b) Let $z=\lambda-(\lambda+1)\mathrm{i}$; $\quad w=\dfrac{\lambda-\lambda\mathrm{i}}{\lambda-(\lambda+1)\mathrm{i}}$	M1
	$=\dfrac{\lambda-\lambda\mathrm{i}}{\lambda-(\lambda+1)\mathrm{i}}\times\dfrac{\lambda+(\lambda+1)\mathrm{i}}{\lambda+(\lambda+1)\mathrm{i}}$	M1
	$u+\mathrm{i}v=\dfrac{\lambda(2\lambda+1)+\lambda\mathrm{i}}{2\lambda^2+2\lambda+1}$	A1
	$u=\dfrac{\lambda(2\lambda+1)}{2\lambda^2+2\lambda+1},\quad v=\dfrac{\lambda}{2\lambda^2+2\lambda+1}$	M1
	$\dfrac{u}{v}=2\lambda+1$	
	$v=\dfrac{2\lambda}{4\lambda^2+4\lambda+2}=\dfrac{(2\lambda+1)-1}{(2\lambda+1)^2+1}=\dfrac{\frac{u}{v}-1}{\left(\frac{u}{v}\right)^2+1}$	M1
	Reducing to the circle with equation $u^2+v^2-u+v=0$ ✱ cso	M1 A1 (7)
	(c) 	
	ft their line	B1ft
	Circle through origin, centre in correct quadrant	B1
	Intersections correctly placed	B1 (3)
		[15]

Question Number	Scheme	Marks
8.	Alternative for (b)	
	Let $z=\lambda-(\lambda+1)\mathrm{i}$; $\quad w=\dfrac{\lambda-\lambda\mathrm{i}}{\lambda-(\lambda+1)\mathrm{i}}$	M1
	$=\dfrac{\lambda-\lambda\mathrm{i}}{\lambda-(\lambda+1)\mathrm{i}}\times\dfrac{\lambda+(\lambda+1)\mathrm{i}}{\lambda+(\lambda+1)\mathrm{i}}$	M1
	$u+\mathrm{i}v=\dfrac{\lambda(2\lambda+1)+\lambda\mathrm{i}}{2\lambda^2+2\lambda+1}$	A1
	$u=\dfrac{\lambda(2\lambda+1)}{2\lambda^2+2\lambda+1}, \quad v=\dfrac{\lambda}{2\lambda^2+2\lambda+1}$	M1
	$u^2+v^2-u+v=\left(\dfrac{\lambda(2\lambda+1)}{2\lambda^2+2\lambda+1}\right)^2+\left(\dfrac{\lambda}{2\lambda^2+2\lambda+1}\right)^2-\dfrac{\lambda(2\lambda+1)}{2\lambda^2+2\lambda+1}+\dfrac{\lambda}{2\lambda^2+2\lambda+1}$	
	$=\dfrac{(4\lambda^4+4\lambda^3+\lambda^2)+\lambda^2-2\lambda^2(2\lambda^2+2\lambda+1)}{(2\lambda^2+2\lambda+1)^2}$	M1
	$=0$ ✱	M1 A1 (7)
8.	Alternative for (b)	
	Let $z=\lambda-(\lambda+1)\mathrm{i}$; $\quad u+\mathrm{i}v=\dfrac{\lambda-\lambda\mathrm{i}}{\lambda-(\lambda+1)\mathrm{i}}$	M1
	$(u+\mathrm{i}v)(\lambda-(\lambda+1)\mathrm{i})=\lambda-\lambda\mathrm{i}$	M1
	$u\lambda+v(\lambda+1)+[v\lambda-u(\lambda+1)]\mathrm{i}=\lambda-\lambda\mathrm{i}$	A1
	Equating real & imaginary parts	
	$u\lambda+v(\lambda+1)=\lambda$ (i) $\quad v\lambda-\lambda u-u=-\lambda$ (ii)	M1
	From (i) $\lambda=\dfrac{v}{1-u-v}$ $\quad$ From (ii) $\lambda=\dfrac{u}{1-u+v}$	
	$\dfrac{v}{1-u-v}=\dfrac{u}{1-u+v}$	M1
	Reducing to the circle with equation $u^2+v^2-u+v=0$ ✱	M1 A1 (7)